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Probability Theory and Related Fields

A Central Limit Theorem for Generalized Quadratic Forms

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Summary. Random variables of the form $W(n)=\sum_{1\leq i\leq n}\sum_{1\leq j\leq n} w_{ijn}(X_i, X_j)$ are considered with X_i independent (not necessarily identically distributed), and $w_{ijn}(\cdot,\cdot)$ Borel functions, such that $w_{ijn}(X_i, X_j)$ is square integrable and has vanishing conditional expectations:

$$E(w_{ijn}(X_i, X_j)\,|\,X_i)=E(w_{ijn}(X_i, X_j)\,|\,X_j)=0, \quad \text{a.s.}$$

A central limit theorem is proved under the condition that the normed fourth moment tends to 3. Under some restrictions the condition is also necessary. Finally conditions on the individual tails of $w_{ijn}(X_i, X_j)$ and an eigenvalue condition are given that ensure asymptotic normality of $W(n)$.

1. Introduction

A simple example of a two parameter process is the quadratic form $a_{ij}X_iX_j$. If the random variables X_i are independent $N(0,1)$ distributed, simple conditions are known that ensure the asymptotic normality of the sum

$$W(n)=\sum_{1\leq i\leq n}\sum_{1\leq j\leq n} a_{ij}X_iX_j.$$

(The matrix (a_{ij}) and the random variables X_i may depend on n, a parameter we suppress.) We assume without loss of generality that the matrix (a_{ij}) is symmetric. Then there is an orthogonal transformation that brings (a_{ij}) into diagonal form and we can rewrite: $W(n)=\sum_{1\leq i\leq n}\mu_i Y_i^2$ with μ_i the eigenvalues of the matrix (a_{ij}) and where the Y_i are $N(0,1)$ distributed, orthogonal and hence independent.

Let the diagonal elements a_{ii} vanish. Then $W(n)=\sum_{1\leq i\leq n}\mu_i(Y_i^2-1)$ is a weighted sum of independent centered chi-square distributed random variables, with total variance $\sigma(n)^2 = 2 \sum_{1 \leqq i \leqq n} \mu_i^2$. Clearly the condition $\sigma(n)^{-2} \max_{1 \leqq i \leqq n} \mu_i^2 \to 0$, is necessary and sufficient for the asymptotic normality of $W(n)$. This condition is equivalent to $\sigma(n)^{-4} \sum_{1 \leqq i \leqq n} \mu_i^4 \to 0$. Straightforward calculation shows that this last condition is equivalent to:

$$\sigma(n)^{-4} E W(n)^4 \to 3, \qquad n \to \infty. \tag{1}$$

In this paper we concentrate on the more general form $W(n) = \sum_{1 \leqq i < j \leqq n} W_{ij}$ with $W_{ij} = w_{ij}(X_i, X_j) + w_{ji}(X_j, X_i)$, where the X_i are independent and $w_{ij}(\cdot, \cdot)$ are Borel measurable such that $E W_{ij}^2 = \sigma_{ij}^2$ is finite, subject to the centering condition that the conditional expectations vanish:

$$E(W_{ij} \mid X_i) = 0 \quad \text{a.s.,} \qquad \text{for all } i, j \leqq n.$$

Theorem 2.1 states that condition (1) is sufficient for the asymptotic normality of $W(n)$ (under the assumption that the variance of each row sum is negligible). The proof (Sect. 3) is quite technical. The centering condition on W_{ij} which plays an important role in this paper, is treated in more detail in Sect. 2. In that section our main results are presented.

It is remarkable that (apart from the negligibility of the row sums) the result for quadratic forms in independent $N(0, 1)$ random variables remains valid in this very general situation. Since the condition on the fourth moment may be hard to check, we give in Sect. 5 simple sufficient conditions which imply (1); the last two theorems extend certain results of Rotar' (1973), respectively Hall (1984).

Theorem 2.3 states that this moment condition comes close to being necessary in the following sense: If $W(n) = \sum_{1 \leqq i < j \leqq n} W_{ij}$ with uniformly bounded sixth moments for the normalized variables $\sigma_{ij}^{-1} W_{ij}$ is asymptotically normal, then $W(n)$ satisfies the moment condition (1). Sect. 4 contains the proof of this result.

If $W(n)$ does not satisfy the above centering condition it can be split into two parts (see Sect. 2). In a forthcoming paper their simultaneous distribution is treated.

We conclude this section with some references. The limit behaviour of the quadratic form in $N(0, 1)$ random variables is treated exhaustively in a short paper by Sevast'yanov (1961). In the little known paper Rotar' (1973), these results are extended to the case with iid random variables with zero mean and finite variance. In Beran (1972) a central limit theorem for quadratic forms is proved using a martingale method, a result related to that in Whittle (1964).

Generalized quadratic forms are a special case of dissociated random variables (McGinley and Sibson (1975)). For a central limit theorem for dissociated random variables in a special case see Noether (1970) and more generally Barbour and Eagleson (1985).

U-statistics have received considerable attention. Here the terms W_{ij} have the form $W_{ij} = w(X_i, X_j)$ where the function $w(\cdot, \cdot)$ is symmetric and does not depend on the indices i, j (but may depend on the suppressed parameter n).

Weber (1983) proves a central limit theorem using a technique based on backward martingales. In Jammalamadaka and Janson (1984) and in Brown and Kildea (1978) central limit theorems are proved using the method of moments. If the above centering condition holds the U-statistic is said to be degenerate. This case is treated in Hall (1984).

The method used by Hall is a generalization of the methods in Beran (1972) and is essentially the same as ours: Under the centering condition the partial sums $U_k = \sum_{1 \leqq j < k} W_{kj}$ form a row of martingale differences with respect to the σ-fields $\sigma(X_1, \ldots, X_k)$. A central limit theorem for martingales can be applied.

Finally we mention Bloemena (1964). This monograph treats quadratic forms with non-independent random variables. The results in Robinson (1985) on weakly exchangeable arrays can be applied to quadratic forms of exchangeable random variables.

2. Statement of the Theorem

Let $X_1, X_2, \ldots$ be independent variables, and let $w_{ijn}(\cdot, \cdot)$ be Borel functions such that $\operatorname{var} w_{ijn}(X_i, X_j)$ is finite. Put

$$W(n) = \sum_{1 \leqq i \leqq n} \sum_{1 \leqq j \leqq n} w_{ijn}(X_i, X_j),$$

and

$$W_{ij} = w_{ijn}(X_i, X_j) + w_{jin}(X_j, X_i).$$

The index n is suppressed in the notation W_{ij}.

Definition 2.1. $W(n)$ is called *clean* if the conditional expectations of W_{ij} vanish:

$$E(W_{ij} \mid X_i) = 0 \quad \text{a.s.} \qquad \text{for all } \; i, j \leqq n.$$

If $W(n)$ is clean, then W_{ij} has zero expectation and the diagonal elements W_{ii} vanish a.s. We shall assume $W_{ii} \equiv 0$. Then $W(n) = \sum_{1 \leqq i < j \leqq n} W_{ij}$. A consequence of Definition 2.1 which will be used frequently in the sequel, is given in the following lemma.

Lemma 2.1. *Let $W(n)$ be clean. Then (under the assumption that the appropriate moments are finite)*

$$E W_{i_1 j_1} W_{i_2 j_2} \ldots W_{i_k j_k} = 0$$

if at least one index has a value occurring only once among $i_1, j_1, i_2, \ldots, i_k, j_k$. Such an index will be called free.

Proof. Assume $i_1 \notin \{j_1, i_2, \ldots, j_k\}$, then

$$\begin{aligned} &E W_{i_1 j_1} W_{i_2 j_2} \ldots W_{i_k j_k} \\ &\quad = E E(W_{i_1 j_1} W_{i_2 j_2} \ldots W_{i_k j_k} \mid X_{j_1}, X_{i_2}, \ldots, X_{j_k}) \\ &\quad = E W_{i_2 j_2} \ldots W_{i_k j_k} E(W_{i_1 j_1} \mid X_{j_1}) = 0. \end{aligned}$$

This proves the lemma.

In fact we have shown more:

$$E(W_{i_1 j_1} \dots W_{i_k j_k} \mid X_{h_1}, \dots, X_{h_r}) = 0 \quad \text{a.s.,} \tag{2}$$

if there is a free index $i \notin \{h_1, \dots, h_r\}$.

A clean random variable can be seen as one term in an orthogonal decomposition. Given a finite sequence $X_1, \dots, X_n$ of independent random variables, any square integrable random variable $Z = Z(X_1, \dots, X_n)$ can be decomposed:

$$Z = EZ + \sum_{1 \leqq i \leqq n} Z_i + \sum_{1 \leqq i < j \leqq n} Z_{ij} + \dots + \sum_{1 \leqq i_1 < \dots < i_k \leqq n} Z_{i_1 \dots i_k} + \dots + Z_{1 \dots n},$$

where the $1 + n + \frac{1}{2}n(n-1) + \dots + 1$ terms are mutually orthogonal, and $Z_{i_1 \dots i_k}$ is determined by: a) it is $X_{i_1}, \dots, X_{i_k}$ measurable, and b) the conditional expectation given any set of $k-1$ variables X_i vanishes.

In this paper we concentrate on the third term $\sum_{1 \leqq i < j \leqq n} Z_{ij}$ in the decomposition. For a detailed account on this decomposition see Karlin and Rinott (1982). We shall only use the following obvious result:

Lemma 2.2. *If $E(Z \mid X_i) = 0$ a.s. $i = 1, \dots, n$, then $Z_{ij} = E(Z \mid X_i, X_j)$ for all $i, j \leqq n$. In particular $Z^* = \sum_{1 \leqq i < j \leqq n} Z_{ij}$ and $Z - Z^*$ are orthogonal and $EZ^{*2} \leqq EZ^2$.*

For a central limit theorem in the case where the variance of the third term is negligible compared to the variance of the second term, see Shapiro and Hubert (1979).

Clean random variables do not only appear as second order approximations: Kester (1975) considers inter-point distances, which are clean by the symmetry of the space.

The main result of this paper is:

Theorem 2.1. *Let $W(n)$ be clean with variance $\sigma(n)^2$. Assume*

a) $\sigma(n)^{-2} \max_{1 \leqq i \leqq n} \sum_{1 \leqq j \leqq n} \sigma_{ij}^2 \to 0, \ n \to \infty.$

b) $\sigma(n)^{-4} EW(n)^4 \to 3, \ n \to \infty.$

Then

$$\sigma(n)^{-1} W(n) \xrightarrow{d} N(0, 1) \qquad n \to \infty.$$

Condition a guarantees that the variance of one individual row sum is negligible compared to the total variance. It rules out forms like $W(n) = \sum_{1 < i \leqq n} X_1 X_i$, which depend crucially on the distribution of X_1. On the other hand this condition is not sufficient to ensure asymptotic normality. The matrix with all off-diagonal entries one has negligible row sums, but it has one large eigenvalue and hence the corresponding quadratic form $\sum_{1 \leqq i < j \leqq n} X_i X_j$ has asymptotically a chi-square distribution.

The proof of Theorem 2.1 rests on three propositions. The proof of the third one is quite technical. If an extra condition is added to those in Theorem 2.1 this proof becomes rather simple. We include it as a separate theorem.

Theorem 2.2. *Let $W(n)$ be clean with variance $\sigma(n)^2$. Assume*

a) $\sigma(n)^{-2} \max_{1 \leqq i \leqq n} \sum_{1 \leqq j \leqq n} \sigma_{ij}^2 \to 0$, $n \to \infty$.

b) $\sigma(n)^{-4} EW(n)^4 \to 3$, $n \to \infty$.

c) *There exists a sequence of real numbers $K(n)$ such that*

$$EW_{ij}^4 \leqq K(n)\, \sigma_{ij}^4 \quad \textit{for all } i, j \leqq n$$

and

$$K(n)\, \sigma(n)^{-2} \max_{1 \leqq i \leqq n} \sum_{1 \leqq j \leqq n} \sigma_{ij}^2 \to 0 \qquad n \to \infty.$$

Then

$$\sigma(n)^{-1}\, W(n) \xrightarrow{d} N(0, 1) \qquad n \to \infty.$$

It is well known that convergence in distribution holds if all moments converge to those of the normal $N(0, 1)$ distribution. In the case of clean random variables (and under condition a) the convergence of the fourth moments is sufficient; if the sixth moment of W_{ij} is of the order of σ_{ij}^6, then convergence of the fourth moment is also necessary. (In the following theorem we restrict only the growth of the sixth normed moment of W_{ij}.)

Theorem 2.3. *Let $W(n)$ be clean with variance $\sigma(n)^2$. Assume*

d) *there exists a sequence of real numbers $K(n)$ such that*

$$EW_{ij}^6 \leqq K(n)\, \sigma_{ij}^6 \quad \textit{for all } i, j \leqq n$$

and

$$K(n)^2\, \sigma(n)^{-2} \max_{1 \leqq i \leqq n} \sum_{1 \leqq j \leqq n} \sigma_{ij}^2 \to 0 \qquad n \to \infty$$

and

$$\sigma(n)^{-1}\, W(n) \xrightarrow{d} N(0, 1) \qquad n \to \infty.$$

Then

$$\sigma(n)^{-4}\, EW(n)^4 \to 3 \qquad n \to \infty.$$

3. The Proofs of Theorem 2.1 and Theorem 2.2

As in Hall (1984) $\sigma(n)^{-1} W(n)$ is written as a sum of martingale differences U_{kn}, with

$$U_{kn} = \sigma(n)^{-1} \sum_{1 \leqq j < k} W_{kj}.$$

U_{kn} is $X_1, \ldots, X_k$ measurable and since $W(n)$ is clean we have

$$E(U_{kn} \mid X_1, \ldots, X_{k-1}) = \sigma(n)^{-1} \sum_{1 \leqq j < k} E(W_{kj} \mid X_j) = 0 \qquad \text{a.s.}$$

To establish the asymptotic normality of $\sigma(n)^{-1} W(n) = \sum_{1 \leqq k \leqq n} U_{kn}$ it is sufficient (see Heyde and Brown, 1970) that

$$\text{I.} \qquad \sum_{1 \leqq k \leqq n} E|U_{kn}|^{2+2\delta} \to 0, \qquad n \to \infty.$$

II. $$E\Big(\sum_{1\leqq k\leqq n} U_{kn}^2 | X_1, \ldots, X_{k-1}\Big) - 1|^{1-\delta} \to 0, \qquad n\to\infty$$

(with $\delta \in (0, 1]$; we take $\delta = 1$).

We shall decompose $EW(n)^4$ into five terms. See Table 1 and Proposition 3.1. In Proposition 3.2 it is shown that if the first four terms (G_{I}, G_{II}, G_{III} and G_{IV}) are small the Conditions I and II above hold. This will enable us to prove the Theorems 2.1 and 2.2.

We introduce the notation:

$$\alpha = \sum_{1\leqq i<j\leqq n} W_{ij}^2$$

$$\beta = \sum_{1\leqq i<j\leqq n} \sum_{1\leqq k\leqq n} W_{ki} W_{kj}$$

$$= \sum_{1\leqq i<j<k\leqq n} (W_{ij} W_{ik} + W_{ji} W_{ju} + W_{ki} W_{kj})$$

$$\gamma = \sum_{1\leqq i<j<k<l\leqq n} (W_{ij} W_{kl} + W_{ik} W_{jl} + W_{il} W_{jk}).$$

Now observe that $W(n)^2 = \alpha + 2\beta + 2\gamma$ and since the W_{ij}'s are uncorrelated we have $EW(n)^2 = E\alpha$.

Proposition 3.1. *Let $W(n)$ be clean. Then the identities summarized in Table 1 hold.*

Proof. The proof is a straightforward calculation. Since $W(n)^2 = \alpha + 2\beta + 2\gamma$,

$$EW(n)^4 = E(\alpha^2 + 4\beta^2 + 4\gamma^2 + 4\alpha\beta + 4\alpha\gamma + 8\beta\gamma).$$

We shall consider the terms in this sum one by one.

Table 1. The table expresses the expectations on the left as linear combinations of the quantities at the top

	G_{I}	G_{II}	G_{III}	G_{IV}	G_{V}
$E\alpha^2$	1	2			2
$E\alpha\beta$			1		
$E\beta^2$		1	2	4	
$E\gamma^2$				2	1
$EW(n)^4$	1	6	12	24	6

$E\alpha^2 = G_{\mathrm{I}} + 2G_{\mathrm{II}} + 2G_{\mathrm{V}}$, etc., where

$$G_{\mathrm{I}} = \sum_{1\leqq i<j\leqq n} EW_{ij}^4$$

$$G_{\mathrm{II}} = \sum_{1\leqq i<j<k\leqq n} (EW_{ij}^2 W_{ik}^2 + EW_{ji}^2 W_{jk}^2 + EW_{ki}^2 W_{kj}^2)$$

$$G_{\mathrm{III}} = \sum_{1\leqq i<j<k\leqq n} (EW_{ij}^2 W_{ki} W_{kj} + EW_{ik}^2 W_{ji} W_{jk} + EW_{kj}^2 W_{ij} W_{ik})$$

$$G_{\mathrm{IV}} = \sum_{1\leqq i<j<k<l\leqq n} (EW_{ij} W_{ik} W_{lj} W_{lk} + EW_{ij} W_{il} W_{kj} W_{kl} + EW_{ik} W_{il} W_{jk} W_{jl})$$

$$G_{\mathrm{V}} = \sum_{1\leqq i<j<k<l\leqq n} (EW_{ij}^2 W_{kl}^2 + EW_{ik}^2 W_{jl}^2 + EW_{il}^2 W_{jk}^2).$$

a) $E\beta\gamma=0=E\alpha\gamma$.

This is a result of Lemma 2.1: each term in γ contains four free indices and each term in β has three indices, consequently each product contains at least one free index and has zero expectation. The same reasoning applies to $E\alpha\gamma$.

b) Lemma 2.1 implies that the general term in $\alpha\beta$, $W_{gh}^2 W_{ki} W_{kj}$ has zero expectation if $\{g,h\}\neq\{i,j\}$. So

$$E\alpha\beta=\sum_{1\leqq i<j\leqq n}\sum_{1\leqq k\leqq n} EW_{ij}^2 W_{ki} W_{kj}=G_{\mathrm{III}}.$$

c) The calculation of $E\alpha^2$ is analogous to that of $W(n)^2$ (*with* W_{ij} replaced by W_{ij}^2); none of the terms contains a free index. Hence

$$E\alpha^2=G_{\mathrm{I}}+2G_{\mathrm{II}}+2G_{\mathrm{V}}.$$

d) All terms in $W(n)^4$ containing five or more different indices have at least one free index and hence zero expectation. This implies

$$E\gamma^2=\sum_{1\leqq i<j<k<l\leqq n} E(W_{ij}W_{kl}+W_{ik}W_{jl}+W_{il}W_{jk})^2=G_{\mathrm{V}}+2G_{\mathrm{IV}}.$$

e) $E\beta^2$ contains terms with three and four different indices. All terms with three different indices are contained in

$$\sum_{1\leqq i<j<k\leqq n} E(W_{ij}W_{ik}+W_{ji}W_{jk}+W_{ki}W_{kj})^2=G_{\mathrm{II}}+2G_{\mathrm{III}}.$$

All terms with four indices that have no free index are contained in

$$\begin{aligned}&\sum_{1\leqq i<j\leqq n} E\Big(\sum_{1\leqq k\leqq n} W_{ki}W_{kj}\Big)^2\\&=G_{\mathrm{II}}+2\sum_{1\leqq i<j\leqq n}\sum_{1\leqq k<l\leqq n} EW_{ki}W_{kj}W_{li}W_{lj}\\&=G_{\mathrm{II}}+4G_{\mathrm{IV}}.\end{aligned}$$

This completes the proof.

The following relation between terms in Table 1 will be used frequently

$$|G_{\mathrm{III}}|\leqq G_{\mathrm{II}},\qquad \text{since}\quad |2W_{ij}W_{ik}|\leqq W_{ij}^2+W_{ik}^2. \tag{3}$$

Proposition 3.2. *Let* $W(n)$ *be clean and let* G_{I}, G_{II} *and* G_{IV} *be of lower order than* $\sigma(n)^4$, *then*

$$\sigma(n)^{-1}W(n)\xrightarrow{d}N(0,1),\qquad n\to\infty.$$

Proof. We shall show that Conditions I and II hold. Condition I follows from

$$\begin{aligned}\sum_{1\leqq k\leqq n} EU_{kn}^4&=\sigma(n)^{-4}\sum_{1\leqq k\leqq n}E\Big(\sum_{1\leqq j<k}W_{kj}\Big)^4\\&=\sigma(n)^{-4}\sum_{1\leqq k\leqq n}E\Big(\sum_{1\leqq j<k}W_{kj}^2+2\sum_{1\leqq i<j<k}W_{ki}W_{kj}\Big)^2\\&=\sigma(n)^{-4}\sum_{1\leqq k\leqq n}\Big(\sum_{1\leqq j<k}EW_{kj}^4+6\sum_{1\leqq i<j<k}EW_{ki}^2W_{kj}^2\Big)\\&\leqq\sigma(n)^{-4}(G_{\mathrm{I}}+6G_{\mathrm{II}})=o(1).\end{aligned}$$

Now we shall prove Condition II. By straightforward calculation we obtain

$$\begin{aligned}
&\operatorname{var}\Big(\sum_{1\leqq k\leqq n} E(U_{kn}^2 \mid X_1, \dots, X_{k-1})\Big)\\
&=\sigma(n)^{-4}\operatorname{var}\Big(\sum_{1\leqq j<k\leqq n} E(W_{kj}^2\mid X_j)\\
&\quad +2\sum_{1\leqq i<j<k\leqq n} E(W_{ki}W_{kj}\mid X_i, X_j)\Big)\\
&=\sigma(n)^{-4}\Big(\operatorname{var}\Big(\sum_{1\leqq j<k\leqq n} E(W_{kj}^2\mid X_j)\Big)\\
&\quad +4\operatorname{var}\Big(\sum_{1\leqq i<j<k\leqq n} E(W_{ki}W_{kj}\mid X_i, X_j)\Big)\Big)\\
&\leqq \sigma(n)^{-4}\Big(G_{\mathrm{I}}+2G_{\mathrm{II}}+4\operatorname{var}\Big(\sum_{1\leqq i<j<k\leqq n} E(W_{ki}W_{kj}\mid X_i, X_j)\Big)\Big)\\
&=o(1).
\end{aligned}$$

The second equality uses orthogonality which follows from

$$E(W_{ki}W_{kj}\mid X_g)=0 \quad \text{a.s. if } i\neq j \text{ for all } g, i, j, k, \tag{4}$$

since the product $W_{ki}W_{kj}$ has a free index unequal g (see (2) after Lemma 2.1).

Equation (4) implies by Lemma 2.2

$$\operatorname{var}\Big(\sum_{1\leqq i<j<k\leqq n} E(W_{ki}W_{kj}\mid X_i, X_j)\Big)\leqq \operatorname{var}\Big(\sum_{1\leqq i<j<k\leqq n} W_{ki}W_{kj}\Big).$$

With $\beta_1=\sum_{1\leqq i<j<k\leqq n}(W_{ij}W_{ik}+W_{ji}W_{jk})$, it remains to show:

$$\operatorname{var}(\beta-\beta_1)=o(\sigma(n)^4). \tag{5}$$

Straightforward calculations give as in Proposition 3.1

$$\operatorname{var}\beta_1=\sum_{1\leqq i<j<k\leqq n} E(W_{ij}^2W_{ik}^2+W_{ji}^2W_{jk}^2+2W_{ij}^2W_{ki}W_{kj})+2G_{\mathrm{IV}}.$$

The first sum is of lower order than $\sigma(n)^4$, since G_{II} is of lower order than $\sigma(n)^4$ (see (3)). Hence, $\operatorname{var}\beta_1=o(\sigma(n)^4)$ by the assumptions of the proposition. And Table 1 gives

$$\operatorname{var}\beta=(G_{\mathrm{II}}+2G_{\mathrm{III}}+4G_{\mathrm{IV}}=)\,o(\sigma(n)^4),$$

which proves (5). This proves the proposition.

Proof of Theorem 2.2. We shall show that the terms G_{I}, G_{II}, G_{III} and G_{IV} are all of lower order than $\sigma(n)^4$.

Condition a implies:

$$\sigma(n)^4=2G_{\mathrm{V}}+o(\sigma(n)^4), \tag{6}$$

since

$$\begin{aligned}
\sigma(n)^4 &= E^2\alpha=\Big(\sum_{1\leqq i<j\leqq n}\sigma_{ij}^2\Big)^2\\
&=2G_{\mathrm{V}}+\sum_{1\leqq i<j\leqq n}\sigma_{ij}^4+2\sum_{1\leqq i<j\leqq n}\sum_{1\leqq k\leqq n}\sigma_{ki}^2\sigma_{kj}^2
\end{aligned}$$

and by the inequality $\sum_{1 \leqq k \leqq n} a_k b_k \leqq (\sum_{1 \leqq k \leqq n} a_k) \max_{1 \leqq k \leqq n} b_k$ for $a_k, b_k \geqq 0$, we have

$$\sum_{1 \leqq i \leqq n} \sum_{1 \leqq j \leqq n} \sum_{1 \leqq k \leqq n} \sigma_{ki}^2 \sigma_{kj}^2$$
$$\leqq (\sum_{1 \leqq i \leqq n} \sum_{1 \leqq k \leqq n} \sigma_{ki}^2) * \max_{1 \leqq k \leqq n} \sum_{1 \leqq j \leqq n} \sigma_{kj}^2 \tag{7}$$

which, by Condition a implies (6).

Using the Cauchy Schwarz inequality in combination with Condition c we obtain with the help of (7):

$$G_{\mathrm{I}} + 2G_{\mathrm{II}} \leqq K(n) \sum_{1 \leqq k \leqq n} (\sum_{1 \leqq j \leqq n} \sigma_{kj}^2)^2$$
$$= o(\sigma(n)^4),$$

which by (3) gives $G_{\mathrm{III}} = o(\sigma(n)^4)$. By (6) and Table 1, Condition b reads:

$$EW(n)^4 - 3\sigma(n)^4 = G_{\mathrm{I}} + 6G_{\mathrm{II}} + 12G_{\mathrm{III}} + 24G_{\mathrm{IV}} + o(\sigma(n)^4)$$
$$= o(\sigma(n)^4).$$

Since the first three terms of the righthand side are of lower order than $\sigma(n)^4$ the same must hold for G_{IV}. By Proposition 3.2 this completes the proof of Theorem 2.2.

Proof of Theorem 2.1. We shall show that the following proposition holds.

Proposition 3.3. *Under the conditions of Theorem 2.1 the terms G_{I}, G_{II} and G_{IV} are all of lower order than $\sigma(n)^4$.*

Proof. From Table 1 and (6) (which holds under Condition a) we have

$$EW(n)^4 - 3\sigma(n)^4 = G_{\mathrm{I}} + 6E\beta^2 + o(\sigma(n)^4).$$

Since both leading terms are non negative we have by Condition b:

$$G_{\mathrm{I}} = o(\sigma(n)^4),$$
$$E\beta^2 = o(\sigma(n)^4). \tag{8}$$

We shall apply the orthogonal decomposition to β and split $E\beta^2$ into two non-negative parts. Since $E(\beta \mid X_g) = 0$ a.s., for all g (see (2)) we have with

$$\beta' = \sum_{1 \leqq i < j \leqq n} E(\beta \mid X_i, X_j)$$
$$E\beta^2 = E\beta'^2 + E(\beta - \beta')^2.$$

With (8) we have

$$E\beta'^2 = o(\sigma(n)^4), \tag{9}$$

$$E(\beta - \beta')^2 = o(\sigma(n)^4). \tag{10}$$

By (2) we have

$$E(\beta \mid X_i, X_j) = \sum_{1 \leqq k \leqq n} E(W_{ki} W_{kj} \mid X_i, X_j),$$

and

$$
\begin{aligned}
E\beta'^2 &= \sum_{1\le i<j\le n} EE^2(\beta \mid X_i, X_j) \\
&= \sum_{1\le i<j\le n} \Big(\sum_{1\le k\le n} EE^2(W_{ki}W_{kj} \mid X_i, X_j) \\
&\quad + 2\sum_{1\le k<l\le n} EE(W_{ki}W_{kj} \mid X_i, X_j)\, E(W_{li}W_{lj} \mid X_i, X_j)\Big) \\
&= o(\sigma(n)^4) + 4G_{\mathrm{IV}}.
\end{aligned}
$$

The last equality sign follows by applying the conditional Cauchy Schwarz inequality to each term in the first sum:

$$
\begin{aligned}
EE^2(W_{ki}W_{kj} \mid X_i, X_j) &\le EE(W_{ki}^2 \mid X_i, X_j)\, E(W_{kj}^2 \mid X_i, X_j) \\
&= EE(W_{ki}^2 \mid X_i)\, E(W_{kj}^2 \mid X_j) \\
&= EW_{ki}^2\, EW_{kj}^2,
\end{aligned}
$$

and by applying the following identity to each term in the second sum:

$$
\begin{aligned}
EW_{ki}W_{kj}W_{li}W_{lj} &= EE(W_{ki}W_{kj}W_{li}W_{lj} \mid X_i, X_j, X_k) \\
&= EW_{ki}W_{kj}E(W_{li}W_{lj} \mid X_i, X_j) \\
&= EE(W_{ki}W_{kj} \mid X_i, X_j)\, E(W_{li}W_{lj} \mid X_i, X_j).
\end{aligned}
$$

This proves (by (9)),

$$G_{\mathrm{IV}} = o(\sigma(n)^4),$$

and by (10),

$$G_{\mathrm{II}} + 2G_{\mathrm{III}} = o(\sigma(n)^4).$$

It remains to estimate G_{II} and G_{III} separately. This can be done with the identities in Table 1 and the Cauchy Schwarz inequality applied to: $G_{\mathrm{III}} = E\alpha\beta$. This finishes the proof of Theorem 2.1.

4. Proof of Theorem 2.3

It suffices to show that the sixth normed moment of $W(n)$ has a uniform bound (see Feller 1971, p. 251). This is shown in Proposition 4.1.

The techniques used in the proof of the proposition are slightly different from those in the preceding paragraph. The estimates are perhaps not the sharpest possible but allow us to reduce the amount of detail that was needed in the proof of Theorem 2.1. Two lemmas precede the proof of Proposition 4.1. (Observe that Lemma 4.1 also holds with $\sigma_{ij}^2 = EW_{ij}^2$, if $W(n)$ is not clean.)

Lemma 4.1. *For $W(n)$ the following inequality holds:*

$$E\,|W_{i_1 j_1} \dots W_{i_k j_k}| \le \sigma_{i_1 j_1} \dots \sigma_{i_k j_k}$$

if each index value occurs exactly twice among $i_1, j_1, \dots, j_k$.

Proof. If each index value occurs exactly twice among the indices the product $W_{i_1 j_1} \dots W_{i_k j_k}$ can be split into independent cyclic products

$$W_{g_1 g_2} W_{g_2 g_3} \dots W_{g_h g_1}, \quad 2 \leqq h \leqq k, \quad (g_1, \dots, g_h) \neq.$$

(Here $(g_1, \dots, g_h) \neq$ denotes $g_1, \dots, g_h$ all mutually unequal.)

If h is even, the cyclic product can be split into two products each containing $h/2$ mutually independent factors W_{ij} and by Cauchy Schwarz:

$$\begin{aligned} &E|W_{g_1 g_2} W_{g_2 g_3} \dots W_{g_h g_1}| \\ &\leqq E^{\frac{1}{2}}(W_{g_1 g_2} W_{g_3 g_4} \dots)^2 E^{\frac{1}{2}}(W_{g_2 g_3} \dots W_{g_h g_1})^2 \\ &= \sigma_{g_1 g_2} \dots \sigma_{g_h g_1}. \end{aligned}$$

If h is odd we have

$$\begin{aligned} &E|W_{g_1 g_2} W_{g_2 g_3} \dots W_{g_h g_1}| \\ &= E|W_{g_1 g_2}| E(|W_{g_2 g_3} \dots W_{g_h g_1}| \,|\, X_{g_1}, X_{g_2}) \\ &\leqq E|W_{g_1 g_2}| E^{\frac{1}{2}}(W_{g_2 g_3}^2 \,|\, X_{g_2}) E^{\frac{1}{2}}(W_{g_h g_1}^2 \,|\, X_{g_1}) * \sigma_{g_3 g_4} \dots \sigma_{g_{h-1} g_h} \\ &\leqq \sigma_{g_1 g_2} \sigma_{g_2 g_3} \dots \sigma_{g_h g_1}, \end{aligned}$$

where the conditional version of the Cauchy Schwarz inequality is applied to $h-1$ factors W_{ij} which gives, combined with the independence of the random variables X_i, the first inequality. The second follows from Cauchy Schwarz. This concludes the proof of Lemma 4.1.

Lemma 4.1 is closely related to the one below.

Lemma 4.2. *For the matrix* (σ_{ij}) *the following inequality holds (with* $\sigma(n)^2 = \sum_{1 \leqq i < j \leqq n} \sigma_{ij}^2$*):*

$$\Sigma_{(g_1, \dots, g_k) \neq} |\sigma_{g_1 g_2} \sigma_{g_2 g_3} \dots \sigma_{g_k g_1}| \leqq \sigma(n)^k.$$

Proof. If $k=2$ equality holds. If $k>2$, even, the product can be split into two products each containing $k/2$ factors σ_{ij} having no indices in common with the other factors in the same product; by Cauchy Schwarz follows:

$$\begin{aligned} &\Sigma_{(g_1, \dots, g_k) \neq} |\sigma_{g_1 g_2} \sigma_{g_2 g_3} \dots \sigma_{g_k g_1}| \\ &\leqq (\Sigma_{(g_1, \dots, g_k) \neq} (\sigma_{g_1 g_2} \sigma_{g_3 g_4} \dots)^2)^{\frac{1}{2}} \\ &* (\Sigma_{(g_1, \dots, g_k) \neq} (\sigma_{g_2 g_3} \dots \sigma_{g_k g_1})^2)^{\frac{1}{2}} \leqq \sigma(n)^k. \end{aligned}$$

The last inequality follows by summing without restriction on the indices. If k is odd the summation is first carried out over $k-2$ indices (each summand contains $(k-1)$ factors σ_{ij}) and Cauchy Schwarz can be applied as above. Then the summation is carried out over each of the two remaining indices each time applying Cauchy Schwarz (the first inequality follows from dropping restrictions on the summation):

$$\begin{aligned} &\Sigma_{(g_1, \dots, g_k) \neq} |\sigma_{g_1 g_2} \sigma_{g_2 g_3} \dots \sigma_{g_k g_1}| \\ &\leqq \Sigma_{(g_1 g_2) \neq} |\sigma_{g_1 g_2}| \Sigma_{(g_3, \dots, g_k) \neq} |\sigma_{g_2 g_3} \dots \sigma_{g_k g_1}| \\ &\leqq \Sigma_{(g_1 g_2) \neq} |\sigma_{g_1 g_2}| (\Sigma_{g_3} \sigma_{g_2 g_3}^2)^{\frac{1}{2}} (\Sigma_{g_k} \sigma_{g_k g_1}^2)^{\frac{1}{2}} * \sigma(n)^{k-3} \\ &\leqq \sigma(n)^k. \end{aligned} \tag{11}$$

This proves Lemma 4.2.

Notice that the proofs of the Lemmas 4.1 and 4.2 run closely parallel: instead of integration over a variable X_i, summation is taken over an index i. Except for the restrictions on the summation, products having no indices in common are independent with respect to the counting measure.

Proposition 4.1. *If $W(n)$ is clean with $EW(n)^2=\sigma(n)^2$ and satisfies:*

a) $$EW_{ij}^6 \leqq K(n)\,\sigma_{ij}^6, \qquad \text{for all } i,j\leqq n \tag{12}$$

b) $$K(n)^2\,\sigma(n)^{-2} \max_{1\leqq i\leqq n} \sum_{1\leqq j\leqq n} \sigma_{ij}^2 \to 0, \qquad n\to\infty \tag{13}$$

then

$$\sigma(n)^{-6}\,EW(n)^6 \leqq C+o(1),$$

with C a constant not depending on n.

Proof. The sixth moment $EW(n)^6$ can be split into several partial sums, in the same way as the fourth moment; the number of partial sums does not depend on n. We shall distinguish these sums according to the number of summation indices. The proof proceeds in two steps. In the first step it is shown that all partial sums with 5 or less indices are of lower order than $\sigma(n)^6$. In the next step the upper bound for the sums with 6 summation indices will be calculated. (Since $W(n)$ is clean, sums with 7 or more indices do not occur in $EW(n)^6$.)

For products with 5 or less indices we obtain by the Hölder inequality and (12):

$$\begin{aligned} E|W_{i_1j_1}\dots W_{i_6j_6}| &\leqq E^{\frac{1}{6}}(W_{i_1j_1})^6\dots E^{\frac{1}{6}}(W_{i_6j_6})^6_6 \\ &\leqq K(n)\,\sigma_{i_1j_1}\dots\sigma_{i_6j_6}. \end{aligned} \tag{14}$$

Consider a product $\sigma_{i_1j_1}\dots\sigma_{i_kj_k}$ without a free index and with $k'<k$ different values among its indices $i_1,j_1,\dots,j_k$. There is at least one index, say i, with a value occurring more than two times. The product then contains a free chain, i.e. a partial product of the form:

$$\sigma_{ig_1}\sigma_{g_1g_2}\dots\sigma_{g_rj} \qquad (0\leqq r\leqq k-2),$$

such that j has a value occurring more than two times (possibly $i=j$) and $g_1,\dots,g_r$ have values occurring exactly twice among $i_1,j_1,\dots,j_k$. The remaining product contains no free index and consists of $k-r-1$ factors σ_{ij} with $k-r$ different index values. After removing $k-k'$ free chains a product remains without a free index and without a free chain. This product contains strictly less than k, say h, factors σ_{ij} (and h different index values).

The sum over all different values for the indices $g_1,\dots,g_r$ of a free chain can be estimated, if $r\geqq 1$, by (11) in Lemma 4.2:

$$\Sigma_{(g_1,\dots,g_r)\neq}|\sigma_{ig_1}\sigma_{g_1g_2}\dots\sigma_{g_rj}| \leqq \sigma(n)^{r-1}\Big(\max_{1\leqq i\leqq n}\sum_{1\leqq j\leqq n}\sigma_{ij}^2\Big),$$

and if $r=0$, by

$$\max_{1\leqq i<j\leqq n}\sigma_{ij} \leqq \Big(\max_{1\leqq i\leqq n}\sum_{1\leqq j\leqq n}\sigma_{ij}^2\Big)^{\frac{1}{2}}.$$

The sum over all different index values of a product of h factors σ_{ij} with each index value occurring exactly twice among the indices can be estimated by $\sigma(n)^h$ (Lemma 4.2). All partial sums in $EW(n)^6$ with 5 or less summation indices contain a free chain, and are by (13) and (14) of lower order than $\sigma(n)^6$. This concludes the first part of the proof.

Now consider the sums with 6 summation indices. If each index value occurs exactly twice among the indices, a sharper inequality than (14) holds (Lemma 4.1):

$$E|W_{i_1 j_1} \dots W_{i_6 j_6}| \leqq \sigma_{i_1 j_1} \dots \sigma_{i_6 j_6}.$$

As in the proof of Lemma 4.1 the righthand side product can be split into cyclic products of the form

$$\sigma_{g_1 g_2} \sigma_{g_2 g_3} \dots \sigma_{g_h g_1}, \qquad \text{with } h=2, 3, 4, 6.$$

By elementary matrix theory we obtain (with $\mu_1, \mu_2, \dots, \mu_n$ the eigenvalues of the symmetric matrix (σ_{ij})):

$$\Sigma_{(g_1, \dots, g_h)} \sigma_{g_1 g_2} \sigma_{g_2 g_3} \dots \sigma_{g_h g_1} = \text{Const} * \sum_{1 \leqq i \leqq n} \mu_i^h.$$

The constant is positive and does not depend on the matrix. Since all matrix elements σ_{ij} are non negative, $\sum_{1 \leqq i \leqq n} \mu_i^h$ is non negative. Dropping the restriction on the summation only alters the sum by $o(\sigma(n)^6)$, as is shown in the first part of the proof. This shows that up to $o(1)$ the sixth normed moment $\sigma(n)^{-6} EW(n)^6$ is bounded by the polynomial $\sigma(n)^{-6}(aM_2^3 + bM_3^2 + cM_4 M_2 + dM_6)$. With a, b, c and d non-negative and not depending on n and $M_h = \sum_{1 \leqq i \leqq n} \mu_i^h$. Since $M_2 = \sigma(n)^2$ one has $\sigma(n)^{-h} M_h \leqq 1$.

This proves Proposition 4.1.

5. Results Involving Only Second Moments

In this section we drop all assumptions on fourth moments. In addition to the usual centering condition, and the existence of second moments we impose a condition on the tails of the distributions of W_{ij}.

Theorem 5.2 is on quadratic forms in independent random variables and contains as a special case the i.i.d. case treated in Rotar' (1971).

It is natural that eigenvalues play an important role in the limiting distribution of a quadratic form. In Theorem 5.1 it is shown that the eigenvalues of the matrix (σ_{ij}) play almost a similar role in the general case.

In Theorem 5.3 we consider a weighted U-statistic, which combines properties of the quadratic form and of U-statistics; as a special case the theorem contains the central limit theorem in Hall (1984).

We start with a lemma that gives a well-known property of eigenvalues. (We shall use G_{IV} in the context of a (non-random) matrix a_{ij} to denote the sum of all terms of the form $a_{ij} a_{ik} a_{lj} a_{lk}$.)

Lemma 5.1. *For the symmetric matrix* (a_{ij}) *with eigenvalues* $\mu_1, \ldots, \mu_n$ *and* $\sum_{1 \leq i \leq n} \mu_i^2 = 1$, *the following two statements are equivalent:*

i) $\max_{1 \leq i \leq n} \mu_i^2 \to 0,\ n \to \infty.$

ii) $\max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2 \to 0$ *and* $G_{\mathrm{IV}} \to 0,\ n \to \infty.$

Proof. Since $\max_{1 \leq i \leq n} \mu_i^4 \leq \sum_{1 \leq i \leq n} \mu_i^4 \leq \max_{1 \leq i \leq n} \mu_i^2$, i) is equivalent to $\sum_{1 \leq i \leq n} \mu_i^4 \to 0$. Straightforward calculations yield (we denote by $a_{ij}^{(k)}$ the ijth element in the kth power of the matrix (a_{ij})):

$$\begin{aligned}\sum_{1 \leq i \leq n} \mu_i^4 &= \operatorname{trace}(a_{ij})^4 = \sum_{1 \leq i \leq n} a_{ii}^{(4)} \\ &= \sum_{1 \leq i \leq n} (a_{ii}^{(2)})^2 + \sum_{1 \leq i \neq j \leq n} (a_{ij}^{(2)})^2 \\ &= \sum_{1 \leq i \leq n} \Big(\sum_{1 \leq j \leq n} a_{ij}^2 \Big)^2 + \sum_{1 \leq i \neq j \leq n} \Big(\sum_{1 \leq k \leq n} a_{ki} a_{kj} \Big)^2.\end{aligned}$$

And the last two terms tend jointly to zero if and only if ii) holds. This proves the lemma.

Now we can formulate the three results:

Theorem 5.1. *Let* $W(n)$ *be clean and let there exist a sequence of real numbers* $K(n)$ *such that:*

1) $$K(n)^2 \sigma(n)^{-2} \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} \sigma_{ij}^2 \to, \qquad n \to \infty$$

and

2) $$\max_{1 \leq i < j \leq n} \sigma_{ij}^{-2} E W_{ij}^2 1_{\{|W_{ij}| > K(n) \sigma_{ij}\}} \to 0, \qquad n \to \infty.$$

If the eigenvalues $\mu_1, \ldots, \mu_n$ *of the matrix* (σ_{ij}) *are negligible:*

3) $$\sigma(n)^{-2} \max_{1 \leq i \leq n} \mu_i^2 \to 0, \qquad n \to \infty$$

then

$$\sigma(n)^{-1} W(n) \xrightarrow{d} N(0, 1), \qquad n \to \infty.$$

Proof. Define the truncated variables:

$$W_{ij}^* = W_{ij} 1_{\{|W_{ij}| \leq K(n) \sigma_{ij}\}}$$

and the clean version of W_{ij}^*:

$$\begin{aligned} W_{ij}' &= W_{ij}^* - E(W_{ij}^* \mid X_i) - E(W_{ij}^* \mid X_j) + E W_{ij}^*, \\ W'(n) &= \sum_{1 \leq i < j \leq n} W_{ij}'. \end{aligned}$$

Notice that $W_{ij} - W_{ij}'$ is the clean version of $W_{ij} - W_{ij}^*$ and by Lemma 2.2 we have

$$\begin{aligned}\operatorname{var}(W(n)-W'(n)) &= \sum_{1\leq i<j\leq n} E(W'_{ij}-W_{ij})^2\\ &\leq \sum_{1\leq i<j\leq n} \sigma_{ij}^2 \big(\max_{1\leq i<j\leq n} \sigma_{ij}^{-2} EW_{ij}^2 1_{\{|W_{ij}|>K(n)\sigma_{ij}\}}\big)\\ &= o(\sigma(n)^2).\end{aligned}$$

Since $W'(n)$ tends to $W(n)$ in L^2 it suffices to check that G'_{I}, G'_{II} and G'_{IV} are $o(\sigma(n)^4)$ by Proposition 3.2. Condition 2 gives:

$$EW'^4_{ij} \leq 16K(n)^2\sigma_{ij}^4;$$

this implies, by Condition 1, that G'_{I}, G'_{II} and G'_{III} are all of lower order than $\sigma(n)^4$. By Lemma 4.1 we have:

$$E|W'_{ij}W'_{ik}W'_{lj}W'_{lk}| \leq \sigma'_{ij}\sigma'_{ik}\sigma'_{lj}\sigma'_{lk} \leq \sigma_{ij}\sigma_{ik}\sigma_{lj}\sigma_{lk}.$$

By Lemma 5.1 and Condition 3 it follows that G_{IV} and G'_{IV} are both of lower order than $\sigma(n)^4$. This completes the proof of Theorem 5.1.

If one applies the above theorem directly to the quadratic form $a_{ij}X_iX_j$ one neglects the signs of the matrix elements a_{ij} (we assume $EX_i^2=1$). The eigenvalues of the matrix (a_{ij}) can be completely different from those of the matrix $(|a_{ij}|)=(\sigma_{ij})$.

If the matrix (a_{ij}) has negligible eigenvalues it has also negligible row sums (Lemma 5.1). In that case Condition 2 below is automatically satisfied if the random variables X_i are i.i.d., as in Rotar' (1971).

Theorem 5.2. *Let* $W(n)=\sum_{1\leq i\neq j\leq n} a_{ij}X_iX_j$ *be a quadratic form in independent random variables* $X_i(EX_i=0,\ EX_i^2=1)$, *with* $\mu_1,\dots,\mu_n$ *the eigenvalues of the symmetric matrix* (a_{ij}), *with vanishing diagonal elements:* $a_{ii}=0$ *for all i. Suppose there exists a sequence of real numbers* $K(n)$ *such that:*

1) $$K(n)^4\sigma(n)^{-2}\max_{1\leq i\leq n}\sum_{1\leq j\leq n} a_{ij}^2 \to 0, \qquad n\to\infty$$

and

2) $$\max_{1\leq i\leq n} EX_i^2 1_{\{|x_i|>K(n)\}} \to 0, \qquad n\to\infty.$$

If the eigenvalues of the matrix (a_{ij}) *are negligible:*

3) $$\sigma(n)^{-2}\max_{1\leq i\leq n}\mu_i^2\to 0, \qquad n\to\infty$$

then

$$\sigma(n)^{-1}W(n)\xrightarrow{d} N(0,1), \qquad n\to\infty.$$

Proof. The proof is similar to that of Theorem 5.1, so we shall omit it, except for one remark on the handling of G_{IV}. Since $EX_i^2=1$ we have for each term in G_{IV}:

$$EW_{ij}W_{ik}W_{lj}W_{lk}=a_{ij}a_{ik}a_{lj}a_{lk}$$

and Condition 3 can be used. Now Theorem 5.2 follows in the same way as Theorem 5.1.

The last theorem is a straightforward generalization of the preceding theorem. The proof is by now obvious and will be omitted (Condition 3b is equivalent to one part of the condition in the central limit theorem in Hall (1984)). Notice that Condition 3b does not involve a condition on the fourth moments of the individual random variables W_{ij}.

Theorem 5.3. *Let X_i be i.i.d. random variables and let for each n the Borel functions $w_n: \mathbb{R}^2 \to \mathbb{R}$ satisfy $Ew_n(X_1, y) = Ew_n(y, X_1) = 0$ for all $y \in R$, and $Ew_n^2(X_1, X_2) = 1$. Let $\mu_{1n}, \ldots, \mu_{nn}$ be the eigenvalues of the symmetric matrix (a_{ijn}) and put $W(n) = \sum_{1 \leq i < j \leq n} a_{ijn} w_n(X_i, X_j)$. Suppose there exists a sequence of real numbers $K(n)$ such that:*

1) $$K(n)^2 \sigma(n)^{-2} \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2 \to 0, \qquad n \to \infty$$

and

2) $$Ew_n^2(X_1, X_2) \, 1_{\{|w_n(X_1 X_2)| > K(n)\}} \to 0, \qquad n \to \infty.$$

Then

$$\sigma(n)^{-1} W(n) \xrightarrow{d} N(0, 1), \qquad n \to \infty$$

if one of the following conditions is true:

3a) $$\sigma(n)^{-2} \max_{1 \leq i \leq n} \mu_{in}^2 \to 0, \qquad n \to \infty.$$

3b) $$Ew_n(X_1, X_2) \, w_n(X_1, X_3) \, w_n(X_4, X_2) \, w_n(X_4, X_3) \to 0, \qquad n \to \infty.$$

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